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Appl. Comput. Harmon. Anal. 25 (2008) 114–132

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Fusion frames and distributed processing

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Received 18 March 2007; revised 15 September 2007; accepted 8 October 2007

Available online 16 October 2007

Communicated by Karlheinz Grochenig

Abstract

Let $\{W_i\}_{i\in I}$ be a (redundant) sequence of subspaces of a Hilbert space each being endowed with a weight v_i, and let $\mathcal{H}$ be the closed linear span of the W_is, a composite Hilbert space. $\{(W_i, v_i)\}_{i\in I}$ is called a *fusion frame* provided it satisfies a certain property which controls the weighted overlaps of the subspaces. These systems contain conventional frames as a special case, however they reach far "beyond frame theory." In case each subspace W_i is equipped with a spanning frame system $\{f_{ij}\}_{j\in J_i}$, we refer to $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ as a *fusion frame system*. The focus of this article is on computational issues of fusion frame reconstructions, unique properties of fusion frames important for applications with particular focus on those superior to conventional frames, and on centralized reconstruction versus distributed reconstructions and their numerical differences. The weighted and distributed processing technique described in this article is not only a natural fit to distributed processing systems such as sensor networks, but also an efficient scheme for parallel processing of very large frame systems. Another important component of this article is an extensive study of the robustness of fusion frame systems.

Keywords: Data fusion; Distributed processing; Frames; Fusion frames; Parallel processing; Sensor networks; Signal reconstruction

1. Introduction

Frames, which are systems that provide robust, stable and usually non-unique representations of vectors, have been a focus of study in the last two decades in applications where redundancy plays a vital and useful role, e.g., filter bank theory [7], sigma–delta quantization [4], signal and image processing [8], and wireless communications [22].

However, a number of new applications have emerged which cannot be modeled naturally by one single frame system. Generally they share a common property that requires distributed processing. Furthermore, we are often

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1 P.G.C. was partially supported by NSF Grant DMS-0704216.
2 G.K. was partially supported by Deutsche Forschungsgemeinschaft (DFG) Research Fellowship KU 1446/5-1, Preis der Justus-Liebig-Universität Gießen 2006, and DFG Heisenberg-Fellowship KU 1446/8-1.
3 S.L. was partially supported by NSF RUI Grant DMS-0406979.

1063-5203/$ – see front matter
doi:10.1016/j.acha.2007.10.001

overwhelmed by a deluge of data assigned to one single frame system, which becomes simply too large to be handled numerically. In these cases it would be highly beneficial to split a large frame system into a set of (overlapping) much smaller systems, and to process locally within each sub-system effectively.

A distributed frame theory for a set of local frame systems is therefore in demand. In this paper we develop a theory based on fusion frames, which provides exactly the framework not only to model these applications but also to derive efficient and robust algorithms.

1.1. Applications under distributed processing requirements

A variety of applications require distributed processing. Among them there are, for instance, wireless sensor networks [23], geophones in geophysics measurements and studies [17], and the physiological structure of visual and hearing systems [28]. To understand the nature, the constraints, and related problems of these applications, let us elaborate a bit further on the example of wireless sensor networks.

In wireless sensor networks, sensors of limited capacity and power are spread in an area sometimes as large as an entire forest to measure the temperature, sound, vibration, pressure, motion, and/or pollutants. In some applications, wireless sensors are placed in a geographical area to detect and characterize chemical, biological, radiological, and nuclear material. Such a sensor system is typically redundant, and there is no orthogonality among sensors, therefore each sensor functions as a frame element in the system. Due to practical and cost reasons, most sensors employed in such applications have severe constraints in their processing power and transmission bandwidth. They often have strictly metered power supply as well. Consequently, a typical large sensor network necessarily divides the network into redundant sub-networks—forming a set of subspaces. The primary goal is to have local measurements transmitted to a local sub-station within a subspace for a subspace combining. An entire sensor system in such applications could have a number of such local processing centers. They function as relay stations, and have the gathered information further submitted to a central processing station for final assembly.

In such applications, distributed/local processing is built in the problem formulation. A staged processing structure is prescribed. We will have to be able to process the information stage by stage from local information and to eventually fuse them together at the central station. We see therefore that a mechanism of coherently collecting sub-station/subspace information is required.

Also, due to the often unpredictable nature of geographical factors, certain local sensor systems are less reliable than others. While facing the task of combining local subspace information coherently, one has also to consider weighting the more reliable sets of substation information more than suspected less reliable ones. Consequently, the coherent combination mechanism we just saw as necessary often requires a weighted structure as well. We will show that fusion frame systems fit such weighted and coherent fusion needs.

1.2. Parallel processing of large frame systems

In case that a frame system is simply too large to handle effectively from the numerical standpoint, there are needs to divide the large system into smaller and parallel ones. Like many parallel processing mechanisms, one may consider splitting the large system into multiple small systems for simpler and parallel processing. It is important for the subdivision mechanism to take into consideration a coherent combination after the subsystem processing. To make the subdivision mechanism more robust, one may not want to (sometimes it is also impossible to) split the large system in an independent or orthogonal fashion. Such a splitting and then a coherent combination must produce precisely the original result if the system were to be processed globally.

Fusion frame systems are created to fit such needs as well. Weighted coherent combination of subsystems (as provided by fusion frame theory) is also useful in such applications where losses of some subsystem information occur. Sometimes, weighted coherent combination is also useful from an efficient and approximation point of view. Some approaches such as the domain decomposition method [33] also use coherent combinations. However, fusion frame theory provides a more flexible framework that also takes local frames into account.

1.3. Fusion frames

In [13], two of the authors studied redundant subspaces for the purpose of easing the construction of frames by building them locally in (redundant) subspaces and then piecing the local frames together by employing a special structure of the set of subspaces. This was referred to as a *frame of subspaces*.

We realized that the idea can be far more reaching than that of building large frames from smaller local ones. The weighted and coherent subspace combination in such a mechanism is exactly what is needed in distributed and parallel processing for many fusion applications as motivated above. We decide on a terminology of *fusion frames* since it reflects much more precisely the essence of the system studied and its applications.

While some basic theory of *fusion frames* is studied in [13], we address in this article issues of fusion frames complementary to that in [13]. More particularly, our focus is on computational issues of fusion frame reconstructions, unique properties of fusion frames important for applications with particular focus on those superior to conventional frames, and centralized reconstruction versus distributed reconstructions and their numerical differences. An extended study on robustness and stability of fusion frames is also presented.

Our study of fusion frame reconstruction will not only provide a comprehensive model for applications which require distributed processing and which employ a distributed structure due to complexity reasons, but also to build efficient algorithms for fusion and reconstruction. We provide a general reconstruction formula by the fusion frame operator and its computational representations, derive a variety of ways to fuse/reconstruct depending on the ability of the application to process off-line or only in real time, and present an iterative algorithm. Since we are also concerned with applications having the choice between distributed and centralized reconstruction we further show that in very special cases those reconstructions are in fact performed by employing the same set of vectors, thereby presenting situations where distributed reconstruction shows the same behavior as centralized reconstruction.

As discussed above, sensor networks in particular suffer significantly from disturbances of individual sensors or even whole sub-networks in the form of, e.g., natural forces. This led us to the study of stability of fusion frame systems not only under perturbations of the subspaces themselves, but even more of the local frame vectors. In order to describe the properties of the affected sensor network explicitly, we present several results which, in particular, give precise estimates for the changes of certain properties of fusion frame systems.

We also observe that fusion frames contain conventional frames as a special case. This theory goes thereby “beyond frame theory.” It turns out that the fusion frame theory is in fact much more delicate due to complicated relations between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to the extreme sensitivity with respect to changes of the weights. This fact will also be illustrated by unveiling a relation between the construction of fusion frames and the famous, still unsolved Kadison–Singer Problem in operator theory from 1959.

Some aspects of the theory of fusion frames have already been applied. Bodmann et al. [6] and Bodmann [5] employed Parseval fusion frames under the term *weighted projective resolution of the identity* for optimal transmission of quantum states and for packet encoding. Also, Rozell et al. [27–29] used fusion frames to study noise reduction in sensor networks and to study overlapping feature spaces of neurons in visual and hearing systems. In [24], Kutyniok et al. have studied fusion frames which are optimally resilient against noise and erasures for random signals and have shown that, surprisingly, these optimal fusion frames are in fact optimal Grassmannian packings.

1.4. Related approaches

We wish to mention that there are related approaches undertaken by Aldroubi et al. [1], and Fornasier [19]. A similar idea was also used by Aldroubi and Gröching in a quite different context in [2]. Some further results on the theory developed in [13] can be found in [3], and an extension was derived by Sun [31,32], however without equipping the subspaces with an underlying structure.

1.5. Contents

The organization of this article is as follows. In Section 2, the definition of fusion frames and fusion frame systems and their fundamental characterization will be given, and connections of fusion frames with conventional frames will be discussed. Classifications of fusion frames are studied in Section 3, thereby deriving a complete characterization of fusion frames and proving uniqueness properties of the fusion frame representation. In Section 4, several fusion frame reconstructions are presented. These are the coherent combinations we discussed earlier. An iterative fusion reconstruction is also constructed in this section. Section 5 is devoted to the robustness of fusion frames, in which the analysis of stability of fusion frame systems to perturbations is extensively carried out.

2. Definition and basic properties of fusion frames

In this section, after briefly recalling the basic definitions and notations of conventional frames, the notion of a *fusion frame* and a *fusion frame system* is introduced. Then we will put our focus on the structure of the fusion frame operator and its connection with the fusion frame bounds. Finally, we will highlight how much more sophisticated the theory of fusion frames is compared to conventional frame theory.

We wish to mention that the definition of a fusion frame and its associated fusion frame operator already appeared in [13] under the label "frame of subspaces." If not mentioned otherwise, all following results are distinctive from those in [13] or are even extensions.

2.1. Review of frames

A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist $0 < A \leqslant B < \infty$ (*lower* and *upper frame bounds*) such that

$$A\|f\|^2 \leqslant \sum_{i\in I} \bigl|\langle f, f_i\rangle\bigr|^2 \leqslant B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{1}$$

The representation space associated with a frame is $\ell_2(I)$. In order to analyze a signal $f \in \mathcal{H}$, i.e., to map it into the representation space, the *analysis operator* $T_{\mathcal{F}}:\mathcal{H} \to \ell_2(I)$ given by $T_{\mathcal{F}} f = \{\langle f, f_i\rangle\}_{i\in I}$ is applied. The associated *synthesis operator*, which provides a mapping from the representation space to $\mathcal{H}$, is defined to be the adjoint operator $T^*_{\mathcal{F}}:\ell_2(I) \to \mathcal{H}$, which is given by $T^*_{\mathcal{F}}(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$. By composing $T_{\mathcal{F}}$ and $T^*_{\mathcal{F}}$ we obtain the *frame operator* $S_{\mathcal{F}}:\mathcal{H} \to \mathcal{H}$, $S_{\mathcal{F}} f = T^*_{\mathcal{F}} T_{\mathcal{F}} f = \sum_{i\in I} \langle f, f_i\rangle f_i$, which is a positive, self-adjoint and invertible operator.

Whenever $\mathcal{F} = \{f_i\}_{i\in I}$ is a frame, we know that there exists at least one *dual frame* $\{\tilde{f}_i\}_{i\in I}$ satisfying

$$f = \sum_{i\in I} \langle f, f_i\rangle \tilde{f}_i = \sum_{i\in I} \langle f, \tilde{f}_i\rangle f_i \quad \text{for all } f \in \mathcal{H}. \tag{2}$$

When $\mathcal{F}$ is a redundant (inexact) frame, there exist infinitely many dual frames $\{\tilde{f}_i\}_{i\in I}$, which can even be characterized [25]. The *canonical dual frame* defined by $\{S^{-1}_{\mathcal{F}} f_i\}_{i\in I}$ is the one having the least square property among all dual frames $\{\tilde{f}_i\}_{i\in I}$, i.e., we have $\sum_{i\in I} |\langle f, S^{-1} f_i\rangle|^2 \leqslant \sum_{i\in I} |\langle f, \tilde{f}_i\rangle|^2$ for all $f \in \mathcal{H}$.

Of particular interest are *A-tight frames*, which are frames whose frame bounds can be chosen as $A = B$ in (1). Provided (1) holds with $A = B = 1$, we call $\mathcal{F}$ a *Parseval frame*. The advantage of working with these frames can be clearly seen by considering the reconstruction formula (2). In these cases the canonical dual frame equals $\{\frac{1}{A} f_i\}_{i\in I}$, and hence we obtain $f = \frac{1}{A} T^*_{\mathcal{F}} T_{\mathcal{F}} f$ for each $f \in \mathcal{H}$, i.e., we can employ the frame elements for both analysis and synthesis. There exist many procedures to construct tight or Parseval frames (cf. [9,14]). However, Parseval frames with special properties are usually particularly difficult to construct, see, e.g., [30].

For more details about the theory and applications of frames we refer the reader to the books by Christensen [16], Daubechies [18], and Mallat [26].

2.2. Fusion frames and fusion frame systems

We will start by stating the definition of a fusion frame.

Definition 2.1. Let I be a countable index set, let $\{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a *fusion frame*, if there exist constants $0 < C \leqslant D < \infty$ such that

$$C\|f\|^2 \leqslant \sum_{i\in I} v_i^2 \bigl\|\pi_{W_i}(f)\bigr\|^2 \leqslant D\|f\|^2 \quad \text{for all } f \in \mathcal{H}, \tag{3}$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. We call C and D the *fusion frame bounds*. The family $\{(W_i, v_i)\}_{i\in I}$ is called a *C-tight fusion frame*, if in (3) the constants C and D can be chosen so that $C = D$, a *Parseval fusion frame* provided that $C = D = 1$, and an *orthonormal fusion basis* if $\mathcal{H} = \bigoplus_{i\in I} W_i$. If $\{(W_i, v_i)\}_{i\in I}$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it a *Bessel fusion sequence* with *Bessel fusion bound* D.

Often it will become essential to consider a fusion frame together with a set of local frames for its subspaces. In this case we will speak of a fusion frame system.

Definition 2.2. Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i, i\in I}$ be a frame for W_i for each $i \in I$. Then we call $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ a *fusion frame system* for $\mathcal{H}$. C and D are the associated *fusion frame bounds* if they are the fusion frame bounds for $\{(W_i, v_i)\}_{i\in I}$, and A and B are the *local frame bounds* if these are the common frame bounds for the *local frames* $\{f_{ij}\}_{j\in J_i}$ for each $i \in I$. A collection of dual frames $\{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$ associated with the local frames will be called *local dual frames*.

To provide a quick inside-look at some intriguing relations between properties of the associated fusion frame and the sequence consisting of all local frame vectors, we present the following theorem from [13] that provides a link between local and global properties. This result will moreover be employed in several proofs in the sequel.

Theorem 2.3. *(See [13, Theorem 3.2].) For each $i \in I$, let $v_i > 0$, let W_i be a closed subspace of $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for W_i with frame bounds A_i and B_i. Suppose that $0 < A = \inf_{i\in I} A_i \leqslant \sup_{i\in I} B_i = B < \infty$. Then the following conditions are equivalent*:

(i) $\{(W_i, v_i)\}_{i\in I}$ *is a fusion frame for* $\mathcal{H}$.
(ii) $\{v_i f_{ij}\}_{j\in J_i, i\in I}$ *is a frame for* $\mathcal{H}$.

In particular, if $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds C and D, then $\{v_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for $\mathcal{H}$ with frame bounds AC and BD. Also if $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame bounds C and D, then $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

Tight frames play a vital role in frame theory due to the fact that they provide easy reconstruction formulas. Tight fusion frames will turn out to be particularly useful for distributed reconstruction as well (cf. Section 4). Notice, that the previous theorem also implies that $\{(W_i, v_i)\}_{i\in I}$ is a C-tight fusion frame for $\mathcal{H}$ if and only if $\{v_i f_{ij}\}_{j\in J_i, i\in I}$ is a C-tight frame for $\mathcal{H}$.

The following result proves that the fusion frame bound C of a C-tight fusion frame can be interpreted as the *redundancy* of this fusion frame.

Proposition 2.4. *Let $\{(W_i, v_i)\}_{i=1}^n$ be a C-tight fusion frame for $\mathcal{H}$ with $\dim \mathcal{H} < \infty$. Then we have*

$$C = \frac{\sum_{i=1}^n v_i^2 \dim W_i}{\dim \mathcal{H}}.$$

Proof. Let $\{e_{ij}\}_{j=1}^{\dim W_i}$ be an orthonormal basis for W_i for each $1 \leqslant i \leqslant n$. By Theorem 2.3 and its implication to tight fusion frames mentioned earlier, the sequence $\{v_i e_{ij}\}_{i=1, j=1}^{n, \dim W_i}$ is a C-tight frame for $\mathcal{H}$. Employing [12, Sec. 2.3] yields that

$$C = \frac{\sum_{i=1}^n \sum_{j=1}^{\dim W_i} \|v_i e_{ij}\|^2}{\dim \mathcal{H}} = \frac{\sum_{i=1}^n v_i^2 \dim W_i}{\dim \mathcal{H}}. \qquad \square$$

We now consider an example which points out some of the difficulties with constructing fusion frames.

Example 2.5. Since almost all applications require a finite model for their numerical treatment, we restrict ourselves to a finite-dimensional space in this example. Suppose $\{f_j\}_{j=1}^N$ is a frame for $\mathbb{R}^M$ with frame bounds A, B. Now we split $\{1, \ldots, N\}$ into K sets $J_1, \ldots, J_K$, and define $W_i = \operatorname{span}\{f_j\}_{j\in J_i}$, $1 \leqslant i \leqslant K$. In the finite-dimensional situation each finite set of vectors forms a frame, in particular $\{f_j\}_{j\in J_i}$ is a frame for W_i for each $1 \leqslant i \leqslant K$. Let C and D be a common lower and upper frame bound, respectively. Theorem 2.3 now implies that $\{(W_i, 1, \{f_j\}_{j\in J_i})\}_{i=1}^K$ is a fusion frame system with fusion frame bounds $\frac{C}{B}$, $\frac{D}{A}$. In order for this process to work effectively, the *local frames* have to possess (uniformly) good lower frame bounds, since these control the computational complexity of reconstruction.

However, it is known [15] that the problem of dividing a frame into a finite number of subsets each of which has good lower frame bounds is equivalent to one of the deepest and most intractable unsolved problems in mathematics: *the 1959 Kadison–Singer Problem*. Therefore, constructing fusion frames in the setting we need will require much more sophisticated methods than trying to merely divide a frame into fusion frame parts.

2.3. Fusion frame operators and their computational aspects

In frame theory an input signal is represented by a collection of scalar coefficients that measure the projection of that signal onto each frame vector. The representation space employed in this theory equals $\ell^2(I)$. However, in fusion frame theory an input signal is represented by a collection of *vector* coefficients that represent the projection (not just the projection energy) onto each subspace. Therefore the representation space employed in this setting is

$$\left(\sum_{i\in I}\bigoplus W_i\right)_{\ell^2} = \bigl\{\{f_i\}_{i\in I} \bigm| f_i \in W_i \text{ and } \bigl\{\|f_i\|\bigr\}_{i\in I} \in \ell^2(I)\bigr\}.$$

We remark that the definition of the analysis, synthesis, and fusion frame operator already appeared in [13].

2.3.1. Definitions

Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. In order to map a signal to the representation space, i.e., to analyze it, the *analysis operator* $T_{\mathcal{W}}$ is employed, which is defined by

$$T_{\mathcal{W}}:\mathcal{H} \to \left(\sum_{i\in I}\bigoplus W_i\right)_{\ell_2} \quad \text{with } T_{\mathcal{W}}(f) = \bigl\{v_i\pi_{W_i}(f)\bigr\}_{i\in I}.$$

It can easily be shown that the *synthesis operator* $T_{\mathcal{W}}^*$, which is defined to be the adjoint operator, is given by

$$T_{\mathcal{W}}^*:\left(\sum_{i\in I}\bigoplus W_i\right)_{\ell_2} \to \mathcal{H} \quad \text{with } T_{\mathcal{W}}^*(f) = \sum_{i\in I} v_i f_i,\ \ f = \{f_i\}_{i\in I} \in \left(\sum_{i\in I}\bigoplus W_i\right)_{\ell_2}.$$

The *fusion frame operator* $S_{\mathcal{W}}$ for $\mathcal{W}$ is defined by

$$S_{\mathcal{W}}(f) = T_{\mathcal{W}}^*T_{\mathcal{W}}(f) = \sum_{i\in I} v_i^2\pi_{W_i}(f).$$

Interestingly, a fusion frame operator exhibits properties similar to a frame operator concerning invertibility. In fact, if $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D, then the associated fusion frame operator $S_{\mathcal{W}}$ is positive and invertible on $\mathcal{H}$, and

$$C\,Id \leqslant S_{\mathcal{W}} \leqslant D\,Id. \tag{4}$$

We refer the reader to [13, Prop. 3.16] for details.

2.3.2. Representation of the fusion frame operator

The fusion frame operator will become essential when employing a fusion frame system for the purpose of distributed fusion/reconstruction (see Section 4). More precisely, the inverse of the fusion frame operator will be employed. Therefore, a further investigation of the fusion frame operator computationally is necessary.

We observe that the fusion frame operator can be expressed in terms of local frame operators as follows.

Proposition 2.6. *Let $\{(W_i, v_i, \mathcal{F}_i = \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$, and let $\widetilde{\mathcal{F}}_i = \{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$ be associated local dual frames. Then the associated fusion frame operator $S_{\mathcal{W}}$ can be written as*

$$S_{\mathcal{W}} = \sum_{i\in I} v_i^2 T_{\widetilde{\mathcal{F}}_i}^* T_{\mathcal{F}_i} = \sum_{i\in I} v_i^2 T_{\mathcal{F}_i}^* T_{\widetilde{\mathcal{F}}_i}.$$

Proof. For all $f \in \mathcal{H}$,

$$S_{\mathcal{W}} f = \sum_{i\in I} v_i^2\pi_{W_i}(f) = \sum_{i\in I} v_i^2 \sum_{j\in J_i}\langle f, f_{ij}\rangle \tilde{f}_{ij} = \sum_{i\in I} v_i^2 \sum_{j\in J_i}\langle f, \tilde{f}_{ij}\rangle f_{ij}.$$

Applying the definition of the analysis operators $T_{\mathcal{F}_i}$, $T_{\tilde{\mathcal{F}}_i}$, and of the associated synthesis operators (see Section 2.1), the result follows immediately. □

2.4. Analysis of the fusion frame bounds

Since the exact values of the fusion frame bounds will be important for determining the rate of convergence for reconstruction algorithms (see Section 4), we will show how to compute the optimal fusion frame bounds. We wish to mention that there exists a well-known similar result for conventional frames.

Theorem 2.7. *Let $\{W_i\}_{i\in I}$ be closed subspaces in $\mathcal{H}$, let $\{v_i\}_{i\in I}$ be positive numbers, and let $S_{\mathcal{W}}$ denote the fusion frame operator associated with $\{(W_i, v_i)\}_{i\in I}$. Then the following conditions are equivalent*:

(i) *$\{(W_i, v_i)\}_{i\in I}$ is a fusion frame with fusion frame bounds C and D.*
(ii) *We have $C\,Id \leqslant S_{\mathcal{W}} \leqslant D\,Id$.*

Moreover, the optimal fusion frame bounds are $\|S_{\mathcal{W}}\|$ and $\|S_{\mathcal{W}}^{-1}\|^{-1}$.

Proof. (i) ⇒ (ii). This is implied by (4).

(ii) ⇒ (i). Let $T_{\mathcal{W}}$ denote the analysis operator associated with $\{(W_i, v_i)\}_{i\in I}$. Since $S_{\mathcal{W}} = T_{\mathcal{W}}^* T_{\mathcal{W}}$ and hence $\|T_{\mathcal{W}}\|^2 = \|S_{\mathcal{W}}\|$, for any $f \in \mathcal{H}$ we obtain

$$\sum_{i\in I} v_i^2 \left\|\pi_{W_i}(f)\right\|^2 = \|T_{\mathcal{W}} f\|^2 \leqslant \|T_{\mathcal{W}}\|^2 \|f\|^2 = \|S_{\mathcal{W}}\| \|f\|^2 \leqslant D\|f\|^2.$$

Also, for all $f \in \mathcal{H}$,

$$\|T_{\mathcal{W}} f\|^2 = \langle T_{\mathcal{W}}^* T_{\mathcal{W}} f, f\rangle = \langle S_{\mathcal{W}} f, f\rangle = \left\langle S_{\mathcal{W}}^{\frac{1}{2}} f, S_{\mathcal{W}}^{\frac{1}{2}} f\right\rangle = \left\|S_{\mathcal{W}}^{\frac{1}{2}} f\right\|^2 \geqslant C\|f\|^2. \qquad \square$$

In conventional frame theory it is relatively easy to generate a variety of different frames with controllable frame operator by applying a self-adjoint, invertible operator T to an initial frame $\mathcal{F}$, since it is well known that $T\mathcal{F}$ is again a frame now endowed with frame operator $TS_{\mathcal{F}}T$. This in particular allows the construction of Parseval frames by applying $T = S_{\mathcal{F}}^{-1/2}$. However, for fusion frames the situation is very different. In fact, the new fusion frame operator is of a different form and, moreover, an extra hypothesis is needed, which will turn out to be quite restrictive.

Proposition 2.8. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with associated fusion frame operator $S_{\mathcal{W}}$, and let T be a self-adjoint and invertible operator on $\mathcal{H}$ satisfying*

$$T^*T(W_i) \subset W_i \quad \textit{for all } i \in I. \tag{5}$$

Then $\{(TW_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame operator $TS_{\mathcal{W}}T^{-1}$.

Proof. For each $i \in I$, let $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ be a frame for W_i with frame operator $S_{\mathcal{F}_i}$. Then $\{Tf_{ij}\}_{j\in J_i}$ is a frame for TW_i with frame operator $TS_{\mathcal{F}_i}T$, since

$$\sum_{j\in J_i} \langle f, Tf_{ij}\rangle Tf_{ij} = T\left(\sum_{j\in J_i} \langle T|_{W_i}^* f, f_{ij}\rangle f_{ij}\right) = TS_{\mathcal{F}_i}Tf \quad \text{for all } f \in W_i.$$

The last equation follows from the hypothesis $T^*T(W_i) \subset W_i$, $i \in I$. Our computation shows that the dual frame of $\{Tf_{ij}\}_{j\in J_i}$ is $\{T^{-1}S_{\mathcal{F}_i}^{-1} f_{ij}\}_{j\in J_i}$.

Then for all $f \in \mathcal{H}$ we compute

$$\sum_{i\in I} v_i^2 \pi_{TW_i}(f) = \sum_{i\in I} v_i^2 \left(\sum_{j\in J_i} \left\langle f, T^{-1}S_{\mathcal{F}_i}^{-1} f_{ij}\right\rangle Tf_{ij}\right) = TS_{\mathcal{W}}T^{-1}f. \qquad \square$$

To analyze hypothesis (5), we derive the following operator-theoretic result.

Proposition 2.9. *Let $T:\mathcal{H}\to\mathcal{H}$ be an operator satisfying $T^*T(W)\subset W$ for all closed $W\subset\mathcal{H}$. Then $T^*T=aI$ for some $a\in\mathbb{R}$. In particular, the only operators having the property that $T|_W$ is self-adjoint for all closed $W\subset\mathcal{H}$ are operators of this form.*

Proof. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. First, for every $f\in\mathcal{H}$, letting $W=\operatorname{span}\{f\}$ and applying our hypothesis yields that $T^*Tf=cf$ for some c.

This implies that, for all $i\in I$ we have $T^*Te_i=a_ie_i$ for some scalar a_i. Now, either $a_i=0$ for all i and hence $T^*T=0I$ or there exists a $j\in I$ so that $a_j\neq 0$. We claim that in the latter case, $a_i\neq 0$ for all $i\in I$. To see this, assume there exists some $i\neq j$ such that $a_i=0$ and set $W=\operatorname{span}\{e_i+e_j\}$. By our hypothesis,

$$T^*T(e_i+e_j)=c(e_i+e_j)=T^*Te_i+T^*Te_j=a_ie_i+a_je_j=a_je_j.$$

This implies $ce_i=(a_j-c)e_j$, hence $c=0$ and therefore $a_je_j=0$, which is a contradiction.

Again, for every $i\neq j$

$$T^*T(e_i+e_j)=c(e_i+e_j)=a_ie_i+a_je_j.$$

Hence, $a_i=a_j=c$, which shows that there exists an a such that $a_i=a$ for all $i\in I$. This finishes the proof. $\square$

This result indeed implies that (5) can rarely be satisfied except when T is some diagonal operator with diagonal entries being of constant absolute value.

Therefore, it will be important to at least obtain some estimates for the fusion frame bounds of a fusion frame of the form $\{(TW_i,v_i)\}_{i\in I}$. For this, we first need a technical lemma, which is taken from [20]. For the convenience of the reader, we provide the short proof.

Lemma 2.10. *(See [20].) Let V be a closed subspace of $\mathcal{H}$ and let T be a bounded operator on $\mathcal{H}$. Then*

$$\pi_VT^*=\pi_VT^*\pi_{\overline{TV}}.$$

Proof. First we observe that $g\in(TV)^\perp=(\overline{TV})^\perp$ implies $T^*g\in V^\perp$. Thus

$$\pi_VT^*(f)=\pi_VT^*(\pi_{\overline{TV}}f)+\pi_VT^*\big((Id-\pi_{\overline{TV}})f\big)=\pi_VT^*\pi_{\overline{TV}}(f). \qquad \square$$

The following result will provide explicit estimates for the fusion frame operator of a fusion frame of the form $\{(TW_i,v_i)\}_{i\in I}$.

Theorem 2.11. *Let $\{(W_i,v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame operator $S_{\mathcal{W}}$ and let T be an invertible operator on $\mathcal{H}$. Then $\{(TW_i,v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame operator $S_{T\mathcal{W}}$ satisfying*

$$\frac{TS_{\mathcal{W}}T^*}{\|T\|^2}\leqslant S_{T\mathcal{W}}\leqslant\big\|T^{-1}\big\|^2T^*S_{\mathcal{W}}T.$$

Proof. Fix $f\in\mathcal{H}$. First we prove the lower bound. Employing Lemma 2.10, we obtain

$$\begin{aligned}\left\langle\frac{TS_{\mathcal{W}}T^*}{\|T\|^2}f,f\right\rangle&=\frac{1}{\|T\|^2}\sum_{i\in I}v_i^2\|\pi_{W_i}T^*f\|^2=\frac{1}{\|T\|^2}\sum_{i\in I}v_i^2\|\pi_{W_i}T^*\pi_{TW_i}f\|^2\\&\leqslant\frac{\|T^*\|^2}{\|T\|^2}\sum_{i\in I}v_i^2\|\pi_{TW_i}f\|^2=\langle S_{T\mathcal{W}}f,f\rangle.\end{aligned}$$

In order to show the upper bound, notice that applying Lemma 2.10 to TW_i and T^{-1} yields $\pi_{TW_i}=\pi_{TW_i}\times T^{*-1}\pi_{W_i}T$. Hence,

$$\begin{aligned}\langle S_{TW}f,f\rangle&=\sum_{i\in I}v_i^2\|\pi_{TW_i}f\|^2=\sum_{i\in I}v_i^2\big\|\pi_{TW_i}T^{*-1}\pi_{W_i}Tf\big\|^2\leqslant\big\|T^{-1}\big\|^2\sum_{i\in I}v_i^2\|\pi_{W_i}Tf\|^2\\&=\big\|T^{-1}\big\|^2\langle S_{\mathcal{W}}Tf,Tf\rangle=\big\langle\big\|T^{-1}\big\|^2T^*S_{\mathcal{W}}Tf,f\big\rangle. \qquad \square\end{aligned}$$

As a corollary we obtain estimates for the optimal fusion frame bounds of a fusion frame $\{(TW_i, v_i)\}_{i\in I}$. Note that this result also appears in [20].

Corollary 2.12. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D, and let T be an invertible operator on $\mathcal{H}$. Then $\{(TW_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds*

$$\frac{C}{\|T^{-1}\|^2\|T\|^2} \quad \text{and} \quad \|T^{-1}\|^2\|T\|^2 D.$$

Proof. By Theorem 2.11,

$$\frac{C}{\|T^{-1}\|^2\|T\|^2} \leqslant \frac{\|TS_{\mathcal{W}}T^*\|}{\|T\|^2} \leqslant \|S_{T\mathcal{W}}\| \leqslant \big\| \|T^{-1}\|^2 T^* S_{\mathcal{W}} T \big\| \leqslant \|T^{-1}\|^2\|T\|^2 D. \qquad \square$$

Specializing to T being the inverse fusion frame operator, we obtain the following result.

Corollary 2.13. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D and fusion frame operator $S_{\mathcal{W}}$. Then $\{(S_{\mathcal{W}}^{-1}W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds $\frac{C^2}{D}$ and $\frac{D^2}{C}$.*

Proof. Letting $T = S_{\mathcal{W}}^{-1}$ in Theorem 2.11 yields

$$\frac{S_{\mathcal{W}}^{-1}}{\|S_{\mathcal{W}}^{-1}\|^2} \leqslant S_{S_{\mathcal{W}}^{-1}\mathcal{W}} \leqslant \|S_{\mathcal{W}}\|^2 S_{\mathcal{W}}^{-1}.$$

Applying (4) finishes the proof. $\square$

2.5. Beyond frame theory

Frames can be shown to be a special case of fusion frames in a particular sense. We will make this precise in the following proposition.

Proposition 2.14. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with frame bounds A, B and frame operator $S_{\mathcal{F}}$. Then $\{(\mathrm{span}\{f_i\}, \|f_i\|)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds A, B and fusion frame operator $S_{\mathcal{F}}$.*

Proof. Observe that for any $f \in \mathcal{H}$,

$$\sum_{i\in I} \|f_i\|^2 \pi_{\mathrm{span}\{f_i\}}(f) = \sum_{i\in I} \|f_i\|^2 \left\langle f, \frac{f_i}{\|f_i\|}\right\rangle \frac{f_i}{\|f_i\|} = \sum_{i\in I} \langle f, f_i\rangle f_i = S_{\mathcal{F}} f.$$

Hence the fusion frame operator for $\{(\mathrm{span}\{f_i\}, \|f_i\|)\}_{i\in I}$ equals $S_{\mathcal{F}}$. Since $\{f_i\}_{i\in I}$ possesses the frame bounds A and B, it follows that $AId \leqslant S_{\mathcal{F}} \leqslant BId$, and $\{(\mathrm{span}\{f_i\}, \|f_i\|)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds A and B. $\square$

In the following remark we highlight the sensitivity of fusion frames to changes of the weights.

Remark 2.15. To demonstrate the richer behavior of fusion frames in contrast to frames, we consider a frame $\mathcal{F} = \{f_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds A, B. Proposition 2.14 implies that $\{(\mathrm{span}\{f_i\}, \|f_i\|)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds A, B and fusion frame operator $S_{\mathcal{F}}$. Since $\{S_{\mathcal{F}}^{-1} f_i\}_{i\in I}$ is the canonical dual frame for $\{f_i\}_{i\in I}$ with bounds B^{-1}, A^{-1} and frame operator $S_{\mathcal{F}}^{-1}$, also $\{(\mathrm{span}\{S_{\mathcal{F}}^{-1} f_i\}, \|S_{\mathcal{F}}^{-1} f_i\|)\}_{i\in I}$ is a fusion frame, but now with fusion frame bounds B^{-1}, A^{-1} and fusion frame operator $S_{\mathcal{F}}^{-1}$. Now it is possible to change the associated weights in order to "move" the associated fusion frame bounds again. This is done by applying Corollary 2.13 to $\{(\mathrm{span}\{f_i\}, \|f_i\|)\}_{i\in I}$, which yields that $\{(S_{\mathcal{F}}^{-1}\,\mathrm{span}\{f_i\}, \|f_i\|)\}_{i\in I} = \{(\mathrm{span}\{S_{\mathcal{F}}^{-1} f_i\}, \|f_i\|)\}_{i\in I}$ is a fusion frame with bounds $\frac{C^2}{D}$ and $\frac{D^2}{C}$. Comparing this fusion frame with $\{(\mathrm{span}\{S_{\mathcal{F}}^{-1} f_i\}, \|S_{\mathcal{F}}^{-1} f_i\|)\}_{i\in I}$ does not only reveal how much more sensitive fusion frames are, but also indicates how critical the selection of the weights can be.

3. Classification of fusion frames

This section is devoted to the study of properties which uniquely determine fusion frames. On the one hand, these results will shed some light on the precise structure of these new objects. On the other hand, they will give insight into possible approaches to explicitly construct fusion frames.

3.1. Characterization of fusion frames in terms of projections

The following theorem gives a characterization of all fusion frames in terms of projections from a larger space. An essential difference to conventional frames consists in the occurrence of special types of projections which "break up" into v_i-isometries on subspaces.

Theorem 3.1. *The following conditions are equivalent*:

(i) $\{(W_i, v_i)\}_{i\in I}$ *is a fusion frame for* $\mathcal{H}$.
(ii) *There exists a Hilbert space* $\mathcal{K} \supset \mathcal{H}$ *with an orthonormal basis* $\{e_{ij}\}_{i\in I,\, j\in J_i}$ *and a* (*non-orthogonal*) *projection* $Q : \mathcal{K} \to \mathcal{H}$ *such that* $Q|_{\operatorname{span}\{e_{ij}\}_{j\in J_i}}$ *is a* v_i*-isometry onto* W_i *for all* $i \in I$.

Proof. (i) $\Rightarrow$ (ii). For each $i \in I$, let $\{\tilde{e}_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. By Theorem 2.3, $\{v_i\tilde{e}_{ij}\}_{i\in I,\, j\in J_i}$ is a frame for $\mathcal{H}$. Hence, by [11], there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ with an orthonormal basis $\{e_{ij}\}_{i\in I,\, j\in J_i}$ and a (non-orthogonal) projection $Q : \mathcal{K} \to \mathcal{H}$ satisfying $Q(e_{ij}) = v_i\tilde{e}_{ij}$ for all $j \in J_i$, $i \in I$. Thus $Q|_{\operatorname{span}\{e_{ij}\}_{j\in J_i}}$ is a v_i-isometry onto W_i for all $i \in I$.

(ii) $\Rightarrow$ (i). Let Q be given as in (ii) and, for each $i \in I$, set $K_i = \operatorname{span}\{e_{ij}\}_{j\in J_i}$, hence $W_i = Q(K_i)$. Further, $\{v_i^{-1}Q(e_{ij}) =: \tilde{e}_{ij}\}_{j\in J_i}$ is an orthonormal basis for W_i for all $i \in I$ and $\{v_i\tilde{e}_{ij}\}_{j\in J_i,\, i\in I}$ is a frame for $\mathcal{H}$. By Theorem 2.3, $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$. $\square$

3.2. Minimality of fusion frame coefficients

The coefficient sequence of the fusion frame decomposition is a vector-valued sequence in $(\sum_{i\in I} \bigoplus W_i)_{\ell_2}$. It is well known that the coefficient sequence of a frame decomposition is precisely the sequence with minimum ℓ_2-norm. Our result shows that also the coefficient sequences of the fusion frame decomposition can be characterized by a quite similar, yet slightly weaker property. The correct norm for our considerations is the ℓ_2-norm for $(\sum_{i\in I} \bigoplus W_i)_{\ell_2}$, which is given by $\|\{g_i\}_{i\in I}\|_2 = (\sum_{i\in I} \|g_i\|^2)^{\frac{1}{2}}$. We further remark that in the sequel the range of an operator will be denoted by Rng.

First, we require a technical lemma, which studies an operator derived from an analysis operator by "interchanging projection and inverse fusion frame operators."

Lemma 3.2. *Let* $\{(W_i, v_i)\}_{i\in I}$ *be a fusion frame for* $\mathcal{H}$ *with fusion frame operator* $S_{\mathcal{W}}$, *let* $T_{S_{\mathcal{W}}^{-1}\mathcal{W}}$ *denote the analysis operator for the fusion frame given by* $\{(S_{\mathcal{W}}^{-1}W_i, v_i)\}_{i\in I}$, *and let* P *be the orthogonal projection of* $(\sum_{i\in I} \bigoplus S_{\mathcal{W}}^{-1}W_i)_{\ell_2}$ *onto* $\operatorname{Rng}(T_{S_{\mathcal{W}}^{-1}\mathcal{W}})$. *Further, let* T *be the operator defined by* $T : \mathcal{H} \to (\sum_{i\in I} \bigoplus S_{\mathcal{W}}^{-1}W_i)_{\ell_2}$, $T(f) = \{v_i S_{\mathcal{W}}^{-1}\pi_{W_i}(f)\}_{i\in I}$. *Then we have*

$$T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}} PT = Id.$$

Proof. For any $f \in \mathcal{H}$, we obtain

$$T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}} T(f) = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1}\pi_{W_i}(f) = S_{\mathcal{W}}^{-1} \sum_{i\in I} v_i^2 \pi_{W_i}(f) = S_{\mathcal{W}}^{-1} S_{\mathcal{W}}(f) = f.$$

Hence,

$$Id = T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}} T = T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}} PT + T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}}\big((Id - P)T\big) = T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}} PT. \qquad \square$$

Now we can prove the following result, which is slightly weaker than the corresponding result for conventional frames, this difficulty resulting from the above examined difference between $T_{S_{\mathcal{W}}^{-1}\mathcal{W}}$ and T. To shed some light on the statement of the next theorem, we mention that the previous lemma yields

$$f = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1}\pi_{W_i}(f) = T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}} P\{v_i S_{\mathcal{W}}^{-1}\pi_{W_i}(f)\}_{i\in I}.$$

Hence the sequence $P\{v_i S_{\mathcal{W}}^{-1}\pi_{W_i}(f)\}_{i\in I}$ does not coincide with the sequence coming from the fusion frame decomposition but is "close" to it.

Theorem 3.3. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame operator $S_{\mathcal{W}}$, let $T_{S_{\mathcal{W}}^{-1}\mathcal{W}}$ denote the analysis operator for the fusion frame given by $\{(S_{\mathcal{W}}^{-1} W_i, v_i)\}_{i\in I}$, and let P be the orthogonal projection of $(\sum_{i\in I}\bigoplus S_{\mathcal{W}}^{-1} W_i)_{\ell_2}$ onto $\mathrm{Rng}(T_{S_{\mathcal{W}}^{-1}\mathcal{W}})$. Then, for every $f\in\mathcal{H}$, we have*

$$\left\| P\{v_i S_{\mathcal{W}}^{-1}\pi_{W_i}(f)\}_{i\in I}\right\|_2 \leqslant \left\|\{v_i g_i\}_{i\in I}\right\|_2$$

for all $\{g_i\}_{i\in I}\in(\sum_{i\in I}\bigoplus S_{\mathcal{W}}^{-1} W_i)_{\ell_2}$ satisfying $f=\sum_{i\in I} v_i^2 g_i$.

Proof. Let $T_{S_{\mathcal{W}}^{-1}\mathcal{W}}$ and $T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}}$ denote the analysis and synthesis operator with respect to the fusion frame $\{S_{\mathcal{W}}^{-1} W_i\}_{i\in I}$, respectively. Let $f\in\mathcal{H}$, and let $\{g_i\}_{i\in I}\in(\sum_{i\in I}\bigoplus S_{\mathcal{W}}^{-1}(W_i))_{\ell_2}$ be given such that $f=\sum_{i\in I} v_i^2 g_i$.

First we decompose $\{v_i g_i\}_{i\in I}$ as

$$\{v_i g_i\}_{i\in I} = \{h_i^{(1)}\}_{i\in I} + \{h_i^{(2)}\}_{i\in I},$$

where $\{h_i^{(1)}\}_{i\in I}\in\mathrm{Rng}(T_{S_{\mathcal{W}}^{-1}\mathcal{W}})$ and $\{h_i^{(2)}\}_{i\in I}\in\mathrm{Rng}(T_{S_{\mathcal{W}}^{-1}\mathcal{W}})^\perp$. Employing the fact that $\{h_i^{(2)}\}_{i\in I}\in\mathrm{Rng}(T_{S_{\mathcal{W}}^{-1}\mathcal{W}})^\perp$, we obtain

$$f = T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}}(\{v_i g_i\}_{i\in I}) = T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}}(\{h_i^{(1)}\}_{i\in I}) + T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}}(\{h_i^{(2)}\}_{i\in I}) = T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}}(\{h_i^{(1)}\}_{i\in I}).$$

Let T be defined as in Lemma 3.2. Since $T^*_{S_{\mathcal{W}}^{-1}\mathcal{W}}$ is one-to-one on the range of T, employing Lemma 3.2 it follows that $\{h_i^{(1)}\}_{i\in I} = PTf$. This in turn implies $\{v_i g_i\}_{i\in I} = P\{v_i S_{\mathcal{W}}^{-1}\pi_{W_i}(f)\}_{i\in I} + \{h_i^{(2)}\}_{i\in I}$. Thus, we finally obtain

$$\left\|\{v_i g_i\}_{i\in I}\right\|_2^2 = \left\| P\{v_i S_{\mathcal{W}}^{-1}\pi_{W_i}(f)\}_{i\in I}\right\|_2^2 + \left\|\{h_i^{(2)}\}_{i\in I}\right\|_2^2. \qquad \square$$

In the special case of Parseval fusion frames, we obtain that the vector-valued sequence coming from the fusion frame decomposition is minimal in the ℓ_2-norm.

Corollary 3.4. *Let $\{(W_i, v_i)\}_{i\in I}$ be a Parseval fusion frame for $\mathcal{H}$, and let $f\in\mathcal{H}$. Then we have*

$$\left\|\{v_i\pi_{W_i}(f)\}_{i\in I}\right\|_2 \leqslant \left\|\{v_i g_i\}_{i\in I}\right\|_2$$

for all $\{g_i\}_{i\in I}$ satisfying $f=\sum_{i\in I} v_i^2 g_i$.

Proof. This follows immediately from Theorem 3.3 by noticing that $\mathrm{Rng}(T^*_{S^{-1}\mathcal{W}}) = \mathrm{Rng}(T)$ in Lemma 3.2. $\square$

4. Distributed fusion/reconstruction

Given a large set of data, some applications such as certain *data fusion problems* [34] require processing the data first locally by employing a frame structure, and then fusing the (computed) subspace information globally. This procedure is called *distributed fusion*, and obviously, the second step can be modeled by employing the framework of fusion frame systems. If the initial data comes from a decomposition of a signal with respect to a global frame such as in *sensor networks problems* [23], and the task consists in precisely reconstructing the initial signal via the procedure mentioned above, we speak of *distributed reconstruction*. In this case we sometimes do have the choice of whether either performing distributed or centralized reconstruction, an issue that will be further elaborated in this section.

4.1. Distributed fusion processing

The first fundamental observation we make consists of the fact that distributed fusion processing is feasible in an elegant way by employing the inverse fusion frame operator.

Proposition 4.1. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame operator $S_{\mathcal{W}}$ and fusion frame bounds C and D. Then we have the reconstruction formula*

$$f = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1}\pi_{W_i}(f) \quad \textit{for all } f \in \mathcal{H}.$$

Proof. Since $S_{\mathcal{W}}$ is invertible, for all $f \in \mathcal{H}$ we have

$$f = S_{\mathcal{W}}^{-1} S_{\mathcal{W}} f = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1}\pi_{W_i}(f). \qquad \square$$

The fusion frame theory in fact provides two different approaches for distributed fusion procedures. For this, let $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$, and let $\{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$ be associated local dual frames. One distributed fusion procedure is from the local projections of each subspace:

$$f = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1}\pi_{W_i} f = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1}\left(\sum_{j\in J_i}\langle f, f_{ij}\rangle \tilde{f}_{ij}\right) \quad \text{for all } f \in \mathcal{H}. \tag{6}$$

In this procedure, the local reconstruction takes place first in each subspace W_i, and the inverse fusion frame operator is applied to each local reconstruction and combined together. Another form of distributed fusion actually acts like a global reconstruction if the coefficients of signal/function decompositions are available:

$$f = \sum_{i\in I} v_i^2 \sum_{j\in J_i}\langle f, f_{ij}\rangle \big(S_{\mathcal{W}}^{-1}\tilde{f}_{ij}\big) \quad \text{for all } f \in \mathcal{H}. \tag{7}$$

The difference in this fusion procedure compared with global frame reconstruction lies in the fact that the (global) dual frame $\{S_{\mathcal{W}}^{-1}\tilde{f}_{ij}\}$ is first calculated at the local level, and then fused into the global dual frame by applying the inverse fusion frame operator. This makes the evaluation of (global) duals much more efficient.

Remark 4.2. Depending on applications, some may require the fusion procedure via (6) such as in sensor networks [23], and geophones in geophysics measurements [17], whereas some may allow for fusion processes via (7) such as parallel processing of large frame systems. Let us examine the orders of computation of the fusion procedures (6) and (7), respectively. Besides the operation of $S_{\mathcal{W}}^{-1}$ in both equations, both fusion procedures have the same number of multiplications. However, (6) typically has less (but real time) inverse fusion frame operations. Specifically, (6) has $|I|$ operations of $S_{\mathcal{W}}^{-1}$ over $|I|$ local reconstructions. On the other hand, (7) requires $\sum_{i\in I}|J_i|$ operations of $S_{\mathcal{W}}^{-1}$ over local dual frames $\{f_{ij}\}_{j\in J_i, i\in I}$, which is typically much larger than the $|I|$ operations in (6). It is nevertheless equally important to point out that the much larger $S_{\mathcal{W}}^{-1}$ operation requirement in (7) can be carried out "off-line," which oftentimes can be advantageous. We would also like to mention that it might be possible to employ methods from domain decomposition such as Schwartz multiplicative and additive alternating algorithms for computing $S_{\mathcal{W}}^{-1}$ "on-line" by using local inversions (cf. [33]).

4.2. Distributed reconstruction and (global) dual frames

The purpose of this section is to study the sequence of vectors employed for distributed reconstruction and to compare distributed with centralized reconstruction. For this, let $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$ with local frame bounds A, B, and let $\{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$ be associated local dual frames. Since, by Theorem 2.3, the sequence $\mathcal{F} = \{v_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for $\mathcal{H}$, we might consider the situation that we are given the (global) frame coefficients $\{\langle f, v_i f_{ij}\rangle\}_{j\in J_i, i\in I}$ of a signal $f \in \mathcal{H}$. For some applications, which do not enforce distributed reconstruction, we might have two ways to reconstruct f. The (global) dual frame $\{S_{\mathcal{F}}^{-1} v_i f_{ij}\}_{j\in J_i, i\in I}$ could be used to perform *centralized reconstruction*, i.e., to compute

$$f = \sum_{i\in I} \sum_{j\in J_i} \langle f, v_i f_{ij}\rangle \big(S_{\mathcal{F}}^{-1} v_i f_{ij}\big).$$

Or, in order to reduce the complexity, we might employ the associated fusion frame operator $S_{\mathcal{W}}$ to perform *distributed reconstruction*, and obtain (compare (7))

$$f = \sum_{i\in I} \sum_{j\in J_i} \langle f, v_i f_{ij}\rangle \big(S_{\mathcal{W}}^{-1} v_i \tilde{f}_{ij}\big).$$

In the sequel we will discuss the difference between the sequences $\{S_{\mathcal{F}}^{-1} v_i f_{ij}\}_{j\in J_i, i\in I}$ and $\{S_{\mathcal{W}}^{-1} v_i \tilde{f}_{ij}\}_{j\in J_i, i\in I}$ in more detail. For now, let us take note of the computational differences. The former requires the computation of the global dual in $\mathcal{H}$ as in conventional frame theory, while the later computes local duals $\{\tilde{f}_{ij}\}_{j\in J_i}$ first in W_i which is generally easier, and then fuse them together to obtain a global dual.

Our first result shows that indeed $\{S_{\mathcal{W}}^{-1} v_i \tilde{f}_{ij}\}_{j\in J_i, i\in I}$ is *a dual frame* for $\mathcal{F}$, but not necessarily the *canonical dual frame*.

Proposition 4.3. *Let $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$ with associated fusion frame operator $S_{\mathcal{W}}$, common local frame bounds and local dual frames $\{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$. Then $\{S_{\mathcal{W}}^{-1} v_i \tilde{f}_{ij}\}_{j\in J_i, i\in I}$ is a dual frame for the frame $\{v_i f_{ij}\}_{j\in J_i, i\in I}$.*

Proof. First we note that $\{v_i f_{ij}\}_{j\in J_i, i\in I}$ indeed forms a frame by Proposition 2.3. Employing the self-adjointness of $S_{\mathcal{W}}$, we have for all $f \in \mathcal{H}$,

$$\sum_{i\in I} \sum_{j\in J_i} \big\langle f, S_{\mathcal{W}}^{-1} v_i \tilde{f}_{ij}\big\rangle v_i f_{ij} = \sum_{i\in I} v_i^2 \sum_{j\in J_i} \big\langle \pi_{W_i}\big(S_{\mathcal{W}}^{-1} f\big), \tilde{f}_{ij}\big\rangle f_{ij} = \sum_{i\in I} v_i^2 \pi_{W_i}\big(S_{\mathcal{W}}^{-1} f\big) = f. \qquad \Box$$

It is interesting to observe that a "dual" relation also holds. We wish to mention that this property does not have quite the same correspondence in conventional frames as well.

Proposition 4.4. *Let $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$ with associated fusion frame operator $S_{\mathcal{W}}$, common local frame bounds and local dual frames $\{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$. Then $\{v_i \tilde{f}_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ and $\{S_{\mathcal{W}}^{-1} v_i f_{ij}\}_{i\in I, j\in J_i}$ is a dual frame for it.*

Proof. The fact that $\{v_i \tilde{f}_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ follows again from Proposition 2.3.

Now using the fact that $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ and $\{S_{\mathcal{W}}^{-1} v_i \tilde{f}_{ij}\}_{i\in I, j\in J_i}$ are a pair of dual frames of $\mathcal{H}$ by Proposition 4.3, we have for all $f \in \mathcal{H}$,

$$f = \sum_{i\in I} \sum_{j\in J_i} \langle f, v_i f_{ij}\rangle S_{\mathcal{W}}^{-1} v_i \tilde{f}_{ij} = S_{\mathcal{W}}^{-1}\left(\sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle f, \tilde{f}_{ij}\rangle f_{ij}\right) = \sum_{i\in I} \sum_{j\in J_i} \langle f, v_i \tilde{f}_{ij}\rangle S_{\mathcal{W}}^{-1} v_i f_{ij}. \qquad \Box$$

In order to compare distributed reconstruction with centralized reconstruction, it is essential to understand when $\{S_{\mathcal{W}}^{-1} v_i \tilde{f}_{ij}\}_{j\in J_i, i\in I}$ equals the canonical dual of the frame $\mathcal{F} = \{v_i f_{ij}\}_{j\in J_i, i\in I}$ (compare also (6)), since in these particular cases distributed and centralized reconstruction coincide. In general, this certainly need not be the case due to the observation that a Parseval fusion frame satisfies $S_{\mathcal{F}} = \sum_{i\in I} S_{\mathcal{F}_i} \pi_{W_i}$ with the $S_{\mathcal{F}_i}$ being the local frame operators. Hence, due to the occurring cross terms, generally we have $\{v_i S_{\mathcal{F}_i}^{-1} f_{ij}\}_{j\in J_i, i\in I} \neq \{v_i S_{\mathcal{F}}^{-1} f_{ij}\}_{j\in J_i, i\in I}$. However, the following result gives some special cases in which distributed and centralized reconstruction indeed coincide.

Proposition 4.5. *Let $\{(W_i, v_i, \mathcal{F}_i = \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$ with associated fusion frame operator $S_{\mathcal{W}}$, common local frame bounds and local dual frames $\{\tilde{f}_{ij}\}_{j\in J_i}$, $i \in I$. If $\{(W_i, v_i)\}_{i\in I}$ is an orthogonal fusion basis or $\{f_{ij}\}_{j\in J_i}$ is a Parseval frame sequence for all $i \in I$, then $\{S_{\mathcal{W}}^{-1} v_i S_{\mathcal{F}_i}^{-1} f_{ij}\}_{j\in J_i, i\in I}$ is the canonical dual frame of the frame $\{v_i f_{ij}\}_{j\in J_i, i\in I}$.*

Proof. By Theorem 2.3, the sequence $\mathcal{F} = \{v_i f_{ij}\}_{j\in J_i, i\in I}$ forms a frame for $\mathcal{H}$, and we denote its frame operator by $S_{\mathcal{F}}$. If $\{(W_i, v_i)\}_{i\in I}$ is an orthogonal fusion basis, then $S_{\mathcal{W}} = Id$ and $S_{\mathcal{F}} = \sum_{i\in I} \bigoplus S_{\mathcal{F}_i}\pi_{W_i}$, and hence $S_{\mathcal{F}}^{-1} = \sum_{i\in I} \bigoplus S_{\mathcal{F}_i}^{-1}\pi_{W_i}$. Provided that $\{f_{ij}\}_{j\in J_i}$ is a Parseval frame sequence, we have $S_{\mathcal{F}_i} = Id$, and we further obtain for all $f \in \mathcal{H}$,

$$S_{\mathcal{W}} f = \sum_{i\in I} v_i^2 \pi_{W_i}(f) = \sum_{i\in I}\sum_{j\in J_i} \langle f, v_i f_{ij}\rangle v_i f_{ij} = S_{\mathcal{F}} f.$$

In both cases the claim follows immediately. □

Finally, we point out a surprising fact, which arises from this result, in the situation of having the subspaces of a fusion frame being spanned by Parseval frames.

Remark 4.6. Let $\{(W_i, v_i, \mathcal{F}_i = \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$ with associated fusion frame operator $S_{\mathcal{W}}$, and let $\mathcal{F}_i$ be Parseval frames for all $i \in I$. By the previous result, the operator $S_{\mathcal{W}}$ is *independent* of the choice of the Parseval frame, since $S_{\mathcal{W}}$ always equals the frame operator of the frame $\{v_i f_{ij}\}_{j\in J_i, i\in I}$. The intuitive reason for this is that provided we take Parseval frames for the subspaces, the frame property of the total collection of frame elements completely mirrors the behavior of the fusion frame.

4.3. Iterative reconstruction

Fusion frame reconstruction can be carried out iteratively as well, just like in frame reconstructions [16]. The specific mechanisms can also be divided in two different ways, depending on whether a local reconstruction actually takes place or not as given in (6) or (7). The method we present refers to the distributed fusion procedure given by (6). This result has a well-known analog for conventional frames [16] with the proof carrying over with small changes, therefore we omit it.

Proposition 4.7. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame in $\mathcal{H}$ with fusion frame operator $S_{\mathcal{W}}$ and fusion frame bounds C, D. Further, let $f \in \mathcal{H}$, and define the sequence $(f_n)_{n\in\mathbb{N}_0}$ by $f_0 = 0$ and $f_n = f_{n-1} + \frac{2}{C+D} S_{\mathcal{W}}(f - f_{n-1})$ for $n \geqslant 1$. Then we have $f = \lim_{n\to\infty} f_n$ with the error estimate*

$$\|f - f_n\| \leqslant \left(\frac{D-C}{D+C}\right)^n \|f\|.$$

Thus every $f \in \mathcal{H}$ can be reconstructed from the fusion frame coefficients $T_{\mathcal{W}}(f) = \{v_i \pi_{W_i}(f)\}_{i\in I}$, since $S_{\mathcal{W}} f$ only requires the knowledge of those coefficients and of the sequence of weights $\{v_i\}_{i\in I}$.

We remark that an application of the Chebyshev method or the conjugate gradient method, as done by Gröchenig [21] for the frame algorithm, should lead to faster convergence.

5. Robustness of fusion frame systems

In this section we analyze the stability of fusion frame systems under perturbations of both the subspaces which constitute a fusion frame and the local frame vectors contained in the subspaces. A motivation for this is that, on the one hand, several complete groups of geophones [17] might be moved to a slightly different location to adjust for transmission conditions, and, on the other hand, in wireless sensor networks the location of single sensors might be changed slightly due to the impact of natural forces [23]. Therefore it is important to study the robustness of fusion frame systems under these two different impacts.

Thus, with these practical aspects in mind, we proceed by first examining perturbations of the subspaces in Section 5.1. Based on those results we then study robustness of a fusion frame system under perturbations of the associated local frames in Section 5.2, which is what we are in particular aiming for.

5.1. Perturbation of the fusion frame

First we would like to point out one fundamental problem with perturbations of fusion frames which is the cause of the substantial technicalities in these results. Since the main ingredients in the definition of a fusion frame are the orthogonal projections onto a set of subspaces, it would be natural to consider perturbations of these projections. However, there does not exist a perturbation of a projection in the sense we would need it. This means, that if P and Q are projections on $\mathcal{H}$, $0 \leqslant \lambda_1, \lambda_2 < 1$, and $\|Pf - Qf\| \leqslant \lambda_1 \|Pf\| + \lambda_2 \|Qf\|$ for all $f \in \mathcal{H}$, then it follows that $P = Q$, which can be easily seen by way of contradiction as follows: If $P \neq Q$, then there exists a vector $f \in \mathcal{H}$ so that $f \perp P(\mathcal{H})$, but also satisfying $Qf \neq 0$ (or vice versa). This yields $\|Pf - Qf\| = \|Qf\| \leqslant \lambda_1 \|Pf\| + \lambda_2 \|Qf\| = \lambda_2 \|Qf\|$, which is a contradiction.

Therefore, we define (λ_1, λ_2)-perturbations of sequences by employing the canonical Paley–Wiener-type definition.

Definition 5.1. Let $\{W_i\}_{i \in I}$ and $\{\widetilde{W}_i\}_{i \in I}$ be closed subspaces in $\mathcal{H}$, let $\{v_i\}_{i \in I}$ be positive numbers, and let $0 \leqslant \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$. If

$$\left\|(\pi_{W_i} - \pi_{\widetilde{W}_i}) f\right\| \leqslant \lambda_1 \|\pi_{W_i} f\| + \lambda_2 \|\pi_{\widetilde{W}_i} f\| + \varepsilon \|f\| \quad \text{for all } f \in \mathcal{H} \text{ and } i \in I,$$

then we say that $\{(\widetilde{W}_i, v_i)\}_{i \in I}$ is a $(\lambda_1, \lambda_2, \varepsilon)$-*perturbation* of $\{(W_i, v_i)\}_{i \in I}$.

Employing this definition, we derive the following result about robustness of fusion frames under small perturbations of the associated subspaces. We remark that a different perturbation result for fusion frames can be derived from [32, Theorem 3.1] by employing a different definition of perturbation, however without weights. Moreover, we would like to point out that this would not lead to a result about robustness of fusion frame systems under disturbances of the local frames, which is what we are in particular aiming for.

Proposition 5.2. *Let $\{(W_i, v_i)\}_{i \in I}$ be a fusion frame for $\mathcal{H}$ with bounds C, D. Choose $0 \leqslant \lambda_1 < 1$ and $\varepsilon > 0$ such that $(1 - \lambda_1)\sqrt{C} - \varepsilon(\sum_{i \in I} v_i^2)^{1/2} > 0$. Further, let $\{(\widetilde{W}_i, v_i)\}_{i \in I}$ be a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of $\{(W_i, v_i)\}_{i \in I}$ for some $0 \leqslant \lambda_2 < 1$. Then $\{(\widetilde{W}_i, v_i)\}_{i \in I}$ is a fusion frame with fusion frame bounds*

$$\left[\frac{(1 - \lambda_1)\sqrt{C} - \varepsilon(\sum_{i \in I} v_i^2)^{1/2}}{1 + \lambda_2}\right]^2 \quad \text{and} \quad \left[\frac{\sqrt{D}(1 + \lambda_1) + \varepsilon(\sum_{i \in I} v_i^2)^{1/2}}{1 - \lambda_2}\right]^2.$$

Proof. We first prove the upper bound. For each $f \in \mathcal{H}$, we obtain

$$\begin{aligned}
\left[\sum_{i \in I} v_i^2 \left\|\pi_{\widetilde{W}_i}(f)\right\|^2\right]^{1/2} &\leqslant \left[\sum_{i \in I} v_i^2 \left(\left\|\pi_{W_i}(f)\right\| + \left\|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\right\|\right)^2\right]^{1/2} \\
&\leqslant \left[\sum_{i \in I} v_i^2 \left(\left\|\pi_{W_i}(f)\right\| + \lambda_1 \left\|\pi_{W_i}(f)\right\| + \lambda_2 \left\|\pi_{\widetilde{W}_i}(f)\right\| + \varepsilon \|f\|\right)^2\right]^{1/2} \\
&\leqslant (1 + \lambda_1)\left[\sum_{i \in I} v_i^2 \left\|\pi_{W_i}(f)\right\|^2\right]^{1/2} + \lambda_2 \left[\sum_{i \in I} v_i^2 \left\|\pi_{\widetilde{W}_i}(f)\right\|^2\right]^{1/2} + \varepsilon \left(\sum_{i \in I} v_i^2\right)^{1/2} \|f\|.
\end{aligned}$$

Now solving for $[\sum_{i \in I} v_i^2 \|\pi_{\widetilde{W}_i}(f)\|^2]^{1/2}$ yields

$$\left[\sum_{i \in I} v_i^2 \left\|\pi_{\widetilde{W}_i}(f)\right\|^2\right]^{1/2} \leqslant \frac{\sqrt{D}(1 + \lambda_1) + \varepsilon(\sum_{i \in I} v_i^2)^{1/2}}{1 - \lambda_2} \|f\|.$$

To prove the lower bound, for all $f \in \mathcal{H}$ we have

$$\begin{aligned}\left[\sum_{i\in I} v_i^2 \|\pi_{\widetilde{W}_i}(f)\|^2\right]^{1/2} &\geqslant \left[\sum_{i\in I} v_i^2\big(\|\pi_{W_i}(f)\| - \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\|\big)^2\right]^{1/2} \\ &\geqslant \left[\sum_{i\in I} v_i^2\big(\|\pi_{W_i}(f)\| - \lambda_1\|\pi_{W_i}(f)\| - \lambda_2\|\pi_{\widetilde{W}_i}(f)\| - \varepsilon\|f\|\big)^2\right]^{1/2} \\ &\geqslant (1-\lambda_1)\left[\sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2\right]^{1/2} - \lambda_2\left[\sum_{i\in I} v_i^2\|\pi_{\widetilde{W}_i}(f)\|^2\right]^{1/2} - \varepsilon\left[\sum_{i\in I} v_i^2\right]^{1/2}\|f\| \\ &\geqslant (1-\lambda_1)\sqrt{C}\|f\| - \lambda_2\left[\sum_{i\in I} v_i^2\|\pi_{\widetilde{W}_i}(f)\|^2\right]^{1/2} - \varepsilon\left[\sum_{i\in I} v_i^2\right]^{1/2}\|f\|.\end{aligned}$$

Solving again for $[\sum_{i\in I} v_i^2\|\pi_{\widetilde{W}_i}(f)\|^2]^{1/2}$ yields

$$\left[\sum_{i\in I} v_i^2 \|\pi_{\widetilde{W}_i}(f)\|^2\right]^{1/2} \geqslant \frac{[(1-\lambda_1)\sqrt{C} - \varepsilon(\sum_{i\in I} v_i^2)^{1/2}]}{1+\lambda_2}\|f\|. \qquad \square$$

5.2. Perturbation of the local frames

The second fundamental problem with perturbing fusion frames locally is that a local perturbation cannot "see" the global structure of the fusion frame and therefore cannot adjust for it. For the notion of perturbations of sequences we employ the canonical Paley–Wiener-type definition (compare [10]):

Definition 5.3. Let $\{f_i\}_{i\in I}$ and $\{\tilde{f}_i\}_{i\in I}$ be sequences in $\mathcal{H}$, and let $0 \leqslant \lambda_1, \lambda_2 < 1$. If

$$\left\|\sum_{i\in I} a_i(f_i - \tilde{f}_i)\right\| \leqslant \lambda_1\left\|\sum_{i\in I} a_i f_i\right\| + \lambda_2\left\|\sum_{i\in I} a_i \tilde{f}_i\right\| \quad \text{for all } \{a_i\}_{i\in I} \in \ell^2(I),$$

then we say that $\{\tilde{f}_i\}_{i\in I}$ is a (λ_1, λ_2)-*perturbation* of $\{f_i\}_{i\in I}$.

First we study the relation between the two subspaces spanned by a sequence and its perturbed version.

Proposition 5.4. *Let $\{f_i\}_{i\in I}$ be a frame sequence in $\mathcal{H}$, and let $0 \leqslant \lambda_1, \lambda_2 < 1$. Suppose that $\{\tilde{f}_i\}_{i\in I}$ is a (λ_1, λ_2)-perturbation of $\{f_i\}_{i\in I}$.*

(i) *Then $\{f_i\}_{i\in I}$ is equivalent to $\{\tilde{f}_i\}_{i\in I}$. In particular, for all $\{a_i\}_{i\in I} \in \ell^2(I)$ we have*

$$\frac{1-\lambda_1}{1+\lambda_2}\left\|\sum_{i\in I} a_i \tilde{f}_i\right\| \leqslant \left\|\sum_{i\in I} a_i f_i\right\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}\left\|\sum_{i\in I} a_i \tilde{f}_i\right\|$$

and $\dim(\operatorname{span}_{i\in I}\{f_i\}) = \dim(\operatorname{span}_{i\in I}\{\tilde{f}_i\})$.

(ii) *Let $W = \operatorname{span}_{i\in I}\{f_i\}$ and $\widetilde{W} = \operatorname{span}_{i\in I}\{\tilde{f}_i\}$. Then*

$$\|\pi_W(\pi_{\widetilde{W}}(f))\| \geqslant \left(\frac{1-\lambda_1}{1+\lambda_2} - \lambda_1\frac{1+\lambda_2}{1-\lambda_1} - \lambda_2\right)\|\pi_{\widetilde{W}}(f)\| \quad \textit{for all } f \in \mathcal{H},$$

i.e., if $\lambda_1, \lambda_2 \leqslant \frac{1}{5}$, then π_W is an isomorphism on $\operatorname{Rng}\pi_{\widetilde{W}}$.

Proof. Part (i) follows from [16].

It remains to prove (ii). For this, let $S_{\widetilde{\mathcal{F}}}$ be the frame operator of $\widetilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$. For $f \in \mathcal{H}$,

$$\begin{aligned}\left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle(f_i-\tilde{f}_i)\right\| &\leqslant \lambda_1\left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle f_i\right\| + \lambda_2\left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle \tilde{f}_i\right\| \\ &\leqslant \lambda_1\frac{1+\lambda_2}{1-\lambda_1}\left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle \tilde{f}_i\right\| + \lambda_2\left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle \tilde{f}_i\right\| \\ &= \left(\lambda_1\frac{1+\lambda_2}{1-\lambda_1}+\lambda_2\right)\left\|\pi_{\widetilde{W}}(f)\right\|.\end{aligned}$$

Employing this relation, we obtain

$$\begin{aligned}\left\|\pi_W\left(\pi_{\widetilde{W}}(f)\right)\right\| &= \left\|\pi_W\left(\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle \tilde{f}_i\right)\right\| \\ &\geqslant \left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle \pi_W(f_i)\right\| - \left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle \pi_W(f_i-\tilde{f}_i)\right\| \\ &\geqslant \frac{1-\lambda_1}{1+\lambda_2}\left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle \tilde{f}_i\right\| - \left\|\pi_W\left(\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle(f_i-\tilde{f}_i)\right)\right\| \\ &\geqslant \frac{1-\lambda_1}{1+\lambda_2}\left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle \tilde{f}_i\right\| - \left\|\sum_{i\in I}\langle f, S_{\tilde{\mathcal{F}}}^{-1}\tilde{f}_i\rangle(f_i-\tilde{f}_i)\right\| \\ &\geqslant \frac{1-\lambda_1}{1+\lambda_2}\left\|\pi_{\widetilde{W}}(f)\right\| - \left(\lambda_1\frac{1+\lambda_2}{1-\lambda_1}+\lambda_2\right)\left\|\pi_{\widetilde{W}}(f)\right\| \\ &= \left(\frac{1-\lambda_1}{1+\lambda_2}-\lambda_1\frac{1+\lambda_2}{1-\lambda_1}-\lambda_2\right)\left\|\pi_{\widetilde{W}}(f)\right\|.\end{aligned}$$

It remains to observe that $\lambda_1, \lambda_2 \leqslant \frac{1}{5}$ implies $\frac{1-\lambda_1}{1+\lambda_2} - \lambda_1\frac{1+\lambda_2}{1-\lambda_1} - \lambda_2 > 0$. $\square$

Remark 5.5. Note that in Proposition 5.4 we did not make use of the frame bounds of $\{f_i\}_{i\in I}$, but only of the constants λ_1, λ_2 associated with the perturbation. Therefore it follows that our argument is symmetric in π_W and $\pi_{\widetilde{W}}$ and that each permutation yields the same bounds.

The following theorem gives a precise statement of how a perturbation of the local frames of a fusion frame system—which certainly results in a perturbation of the associated fusion frame—affects its fusion frame bounds.

Theorem 5.6. *Let $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system for $\mathcal{H}$ with fusion frame bounds C, D. Choose $0 \leqslant \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$ such that $1 - \frac{\varepsilon^2}{2} = (\frac{1-\lambda_1}{1+\lambda_2} - \lambda_1\frac{1+\lambda_2}{1-\lambda_1} - \lambda_2)$ and $\sqrt{C} - \varepsilon(\sum_{i\in I} v_i^2)^{1/2} > 0$. For every $i \in I$, let $\{\tilde{f}_{ij}\}_{j\in J_i}$ be a (λ_1, λ_2)-perturbation of $\{f_{ij}\}_{j\in J_i}$ and let $\widetilde{W}_i = \operatorname{span}\{\tilde{f}_{ij}\}_{j\in J_i}$. Then $\{(\widetilde{W}_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds*

$$\left[\sqrt{C} - \varepsilon\left(\sum_{i\in I} v_i^2\right)^{1/2}\right]^2 \quad \textit{and} \quad \left[\sqrt{D} + \varepsilon\left(\sum_{i\in I} v_i^2\right)^{1/2}\right]^2.$$

Proof. Fix $i \in I$. Recalling Proposition 5.4, for all $f \in \mathcal{H}$ we have

$$\left\|\pi_{W_i}(f)\right\|^2 = \left\|\pi_{\widetilde{W}_i}\pi_{W_i}(f)\right\|^2 + \left\|(I-\pi_{\widetilde{W}_i})\pi_{W_i}(f)\right\|^2 \geqslant \left(1-\frac{\varepsilon^2}{2}\right)\left\|\pi_{W_i}(f)\right\|^2 + \left\|(I-\pi_{\widetilde{W}_i})\pi_{W_i}(f)\right\|^2.$$

Hence, $\|(I-\pi_{\widetilde{W}_i})\pi_{W_i}(f)\|^2 \leqslant \frac{\varepsilon^2}{2}\|\pi_{W_i}(f)\|^2$. Employing Remark 5.5, Proposition 5.4 also yields $\|(I-\pi_{W_i})\pi_{\widetilde{W}_i}(f)\|^2 \leqslant \frac{\varepsilon^2}{2}\|\pi_{\widetilde{W}_i}(f)\|^2$. Collecting the estimates derived above, for any $f \in \mathcal{H}$ we obtain

$$\begin{aligned}\big\|(\pi_{W_i}-\pi_{\widetilde{W}_i})(f)\big\|^2 = \big\langle(\pi_{W_i}-\pi_{\widetilde{W}_i})^2(f), f\big\rangle &= \big\langle(\pi_{W_i}-\pi_{\widetilde{W}_i}\pi_{W_i}+\pi_{\widetilde{W}_i}-\pi_{W_i}\pi_{\widetilde{W}_i})(f), f\big\rangle \\ &\leqslant \big\|(I-\pi_{\widetilde{W}_i})\big(\pi_{W_i}(f)\big)+(I-\pi_{W_i})\big(\pi_{\widetilde{W}_i}(f)\big)\big\|\|f\| \\ &\leqslant \frac{\varepsilon^2}{2}\big\|\pi_{W_i}(f)\big\|\|f\|+\frac{\varepsilon^2}{2}\big\|\pi_{\widetilde{W}_i}(f)\big\|\|f\| \\ &\leqslant \varepsilon^2\|f\|^2.\end{aligned}$$

The theorem now follows from Proposition 5.2. □

Acknowledgments

Parts of the research for this paper was performed while the second author was visiting the Department of Mathematics at the University of Missouri in Columbia. This author thanks this department for its hospitality and support during this visit. The authors are very thankful to J.C. Tremain for very helpful suggestions concerning the notion of perturbation for fusion frames which improved the paper. Further, we would like to thank C. Heil for valuable comments on an earlier version of this paper. We are also indebted to H. Bölcskei, H. Feichtinger, M. Fornasier, J. Tanner, and K. Vasudevan for interesting discussions concerning further applications of fusion frames. We would further like to thank the anonymous referees for their valuable comments and suggestions.

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